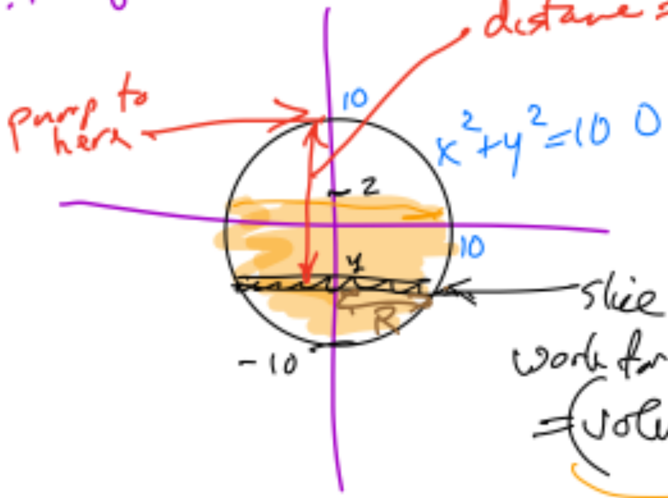
7.17 question

distance $= 10 - y$.



Work for slice

$= \underbrace{(\text{volume of slice})(\text{density}) \cdot g}_{\text{weight of slice}}$

$\cdot (\text{distance of slice})$

$R = x - 0 = x = \sqrt{100 - y^2}$

$= (\pi R^2\, dy)\left(1 \frac{gm}{cm^3}\right)\left(\frac{9.80\cdots m}{s^2}\right)$

$\cdot (10 - y)$ ← distance

Total Work $= \displaystyle\int_{-10}^{2} (\;\text{"}\;) \,.$

$\frac{1\,gm}{cm^3} = \frac{10^{-3}\,kg}{(10^{-2}\,m)^3} = \frac{10^{-3}\,kg}{10^{-6}\,m^3} = \frac{10^3\,kg}{m^3}$

$\pi(100 - y^2)(1)(9.8\ldots) \cdot 10^3 \cdot (10-y)\, dy$

Sequences & Series

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots$$

$$S_N = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^N}$$

$$S_N = \sum_{k=0}^{N} \frac{1}{2^k}$$

$$S_0 = \frac{1}{2^0} = 1$$

$$S_1 = \frac{1}{2^0} + \frac{1}{2^1} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_2 = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$S_3 = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{15}{8}$$

Notice

$$S_N = \frac{2^{N+1} - 1}{2^N}$$

$$S_\infty = \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{2^{N+1}-1}{2^N}$$

$$= \lim_{N\to\infty} \left(\frac{2^{N+1}}{2^N} - \frac{1}{2^N}\right)$$

$$= \lim_{N\to\infty} \left(2 - \underbrace{\frac{1}{2^N}}_{\to 0}\right) = \boxed{2}.$$

Sequences: functions from $\mathbb{N} \to \mathbb{R}$

$$\mathbb{N} = \{\underset{n}{1}, \underset{n}{2}, 3, 4, 5, \ldots\}.$$

We write a sequence often like this

$$(x_1, x_2, x_3, \ldots)$$

x_n = some formula.

Example: Let $x_k = k^2 + 1$.

Find the first 4 terms of the sequence.

Solution: $x_1 = 1^2+1 = 2$ $\quad$ $x_4 = 4^2+1 = 17$.

$x_2 = 2^2+1 = 5$

$x_3 = 3^2+1 = 10$,

This sequence is $(2, 5, 10, 17, \dots)$

This example is an example of a closed-form formula for a sequence.

i.e. $x_k = (\text{function of } k)$ for $k \geq 1$

Other examples: $y_k = k^k - 2k + 1$ for $k \geq 1$

$a_k = 5 - k$ for $k \geq 0$

$(y_k) = (0, 1, 22, \dots)$

$(a_k) = (5, 4, 3, 2, 1, 0, -1, \dots)$

Another method of defining a sequence: Recursive formula:

$x_n = (\text{function of } n \text{ and } x_j \text{ for } j < n)$

Example: $x_0 = 4$, $x_n = x_{n-1} + 3$ for $n \geq 1$

recursive

$x_0 = 4$, $x_1 = x_0 + 3 = 4 + 3 = 7$,

$x_2 = x_1 + 3 = 7 + 3 = 10$

$$(x_n) = (4, 7, 10, 13, 16, \ldots)$$

Our first example:

$$x_k = k^2 + 1 \quad \text{for } k \geq 1$$

$$(\overset{x_1}{2}, \overset{x_2}{5}, \overset{x_3}{10}, \overset{x_4}{17}, \ldots)$$

could be defined in a different way:

$$\begin{cases} x_1 = 2 & \text{(initial condition)} \\ x_n = x_{n-1} + 2n - 1 & \text{(recursive formula)} \end{cases}$$

$x_2 = x_1 + 4 - 1$

$x_3 = x_2 + 6 - 5$

Computing limits of sequence (if they exist).

Ex Let $x_k = \dfrac{k^2}{(k-1)(k+1)}$ for $k \geq 2$.

Find $\lim\limits_{k \to \infty} x_k$.

$$\lim_{k\to\infty} \frac{k^2}{(k-1)(k+1)} = \lim \frac{k^2}{k^2-1}$$

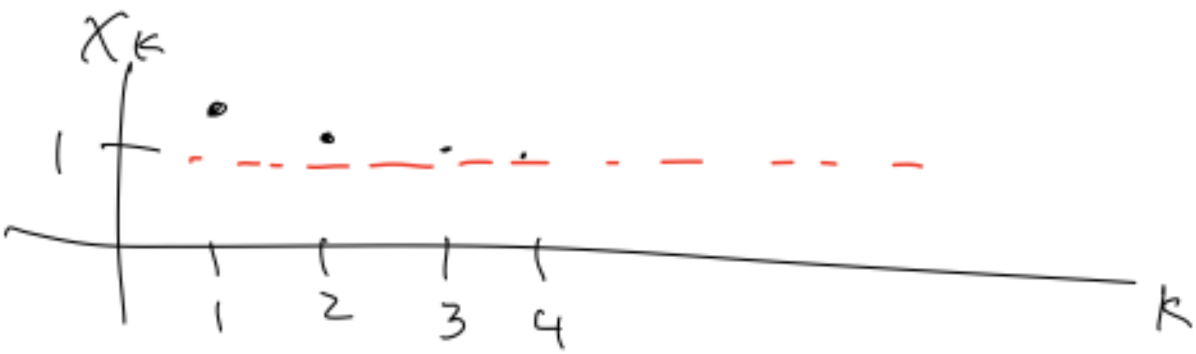
$$= \lim_{k\to\infty} \frac{\cancel{k^2}\ 1}{\cancel{k^2}\left(1-\frac{1}{k^2}\right)}$$

$$= \lim_{k\to\infty} \frac{1}{1-\underbrace{1/k^2}_{0}} = \boxed{1}$$

$$x_k = \left(\frac{4}{1\cdot 3}, \frac{9}{2\cdot 4}, \frac{16}{3\cdot 5}, \frac{25}{4\cdot 6}, \dots\right)$$

Note $x_k > 1$ for every k, but

$$\lim_{k\to\infty} x_k = 1.$$



Example Let $a_m = 1 + \dfrac{1}{a_{m-1}}$ for $m \geq 2$

$a_1 = 1.$

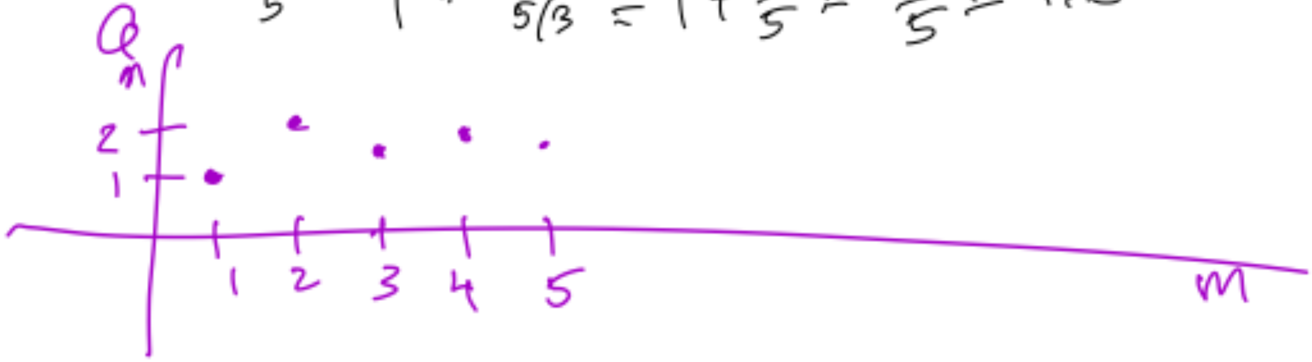
Find $\lim_{m\to\infty} a_m.$

Explore: $a_1 = 1$, $a_2 = 1 + \frac{1}{1} = 2$

$$a_3 = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$$

$$a_4 = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \frac{5}{3} = 1.666\ldots$$

$$a_5 = 1 + \frac{1}{5/3} = 1 + \frac{3}{5} = \frac{8}{5} = 1.6$$



You can find limits of recursive sequences like this: take limit of both sides (assume $\lim a_m = L$).

Recursive formula: $$a_m = 1 + \frac{1}{a_{m-1}}$$

$$\lim_{m \to \infty} a_m = \lim_{m \to \infty} \left(1 + \frac{1}{a_{m-1}}\right)$$

$$\Rightarrow \lim_{m \to \infty} a_m = 1 + \frac{1}{\lim_{m \to \infty} a_{m-1}}$$ ☆

by the algebraic properties of limits.

Assume $L = \lim_{n\to\infty} a_n$

$n \geq 2$ $\quad a_n = (a_2, a_3, a_4, \dots) \to$ converges to L

$a_{n-1} = (a_1, a_2, a_3, \dots) \to$ converges to L also.

$\Rightarrow \lim_{n\to\infty} a_{n-1} = L$ also.

Using $\star$, $\quad L = 1 + \dfrac{1}{L}$

if $L = \lim_{n\to\infty} a_n$

let's try to solve for L:

mult by L $\quad L^2 = L + 1$

$$L^2 - L - 1 = 0$$

$$\Rightarrow L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2} \quad \therefore L = \frac{1-\sqrt{5}}{2} \text{ or } \frac{1+\sqrt{5}}{2}$$

From experimenting, it looks like the limit is converging to something between 1.6 & 1.7

$$\frac{1+\sqrt{5}}{2} = 1.61\ldots \qquad \frac{1-\sqrt{5}}{2} = -.61\ldots$$

We expect the limit to be $\boxed{\frac{1+\sqrt{5}}{2}}$.

Let's examine this with software.

Type some Sage code below and press Evaluate.

```
a=[1]
for k in range(2,20): #range(2,6) means 2,3,4,5
    p=a[-1] #means the last element of the list a
    new=1+1/p
    a += [new];

show(a)
anum=[b.n() for b in a];
show(anum)
show(((1+sqrt(5))/2).n())
```

Evaluate

$\left[1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \frac{233}{144}, \frac{377}{233}, \frac{610}{377}, \frac{987}{610}, \frac{1597}{987}, \frac{2584}{1597}, \frac{4181}{2584}, \frac{676}{418}\right.$

[1.00000000000000, 2.00000000000000, 1.50000000000000, 1.66666666666667, 1.600000000

1.61803398874989

Notice if we keep the same recursive formula but change the initial condition, we could get a different limit. $a_n = 1 + \frac{1}{a_{n-1}}$ for $n \geq 2$

e.g. $a_1 = \frac{1-\sqrt{5}}{2}$

$$a_2 = 1 + \frac{1}{\left(\frac{1-\sqrt{5}}{2}\right)} = 1 + \frac{2}{1-\sqrt{5}}$$

$$= 1 + \frac{2\,(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})}$$

$$= 1 + \frac{2+2\sqrt{5}}{1-5} = 1 + \frac{2+2\sqrt{5}}{-4}$$

$$= \frac{4-2-2\sqrt{5}}{4} = \frac{2-2\sqrt{5}}{4}$$

$$= \frac{1-\sqrt{5}}{2}.$$

$$a_3 = 1 + \frac{1}{a_2} = \frac{1-\sqrt{5}}{2}$$

$$a_4 = \frac{1-\sqrt{5}}{2}.$$

Sequence: $-.618 \quad -.618$

$$\left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

so $\lim\limits_{m\to\infty} a_m = \frac{1-\sqrt{5}}{2}$.

After investigating with different values of a_1, the recursive formula

$$a_m = 1 + \frac{1}{a_{m-1}}$$

results in a sequence that converges to $\frac{1+\sqrt{5}}{2}$ if $a_1 \neq \frac{1-\sqrt{5}}{2}$

& converges to $\frac{1-\sqrt{5}}{2}$ if $a_1 = \frac{1-\sqrt{5}}{2}$.

The limits $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ are called <u>equilibrium points</u> of the dynamical system $a_m = 1 + \frac{1}{a_{m-1}}$.

$\frac{1+\sqrt{5}}{2}$ is a stable equilibrium point.

(if a_1 is close to $\frac{1+\sqrt{5}}{2}$, the sequence will converge to $\frac{1+\sqrt{5}}{2}$)

$\frac{1-\sqrt{5}}{2}$ is an unstable equilibrium point

(if a_1 is close to $\frac{1-\sqrt{5}}{2}$ but not equal to $\frac{1-\sqrt{5}}{2}$, the sequence will not converge to $\frac{1-\sqrt{5}}{2}$.)

Examples Ⓐ Closed-form formula for sequence.

Ⓑ Recursive formula for the sequence.

① $\left(\underset{A_1}{\frac{3}{7}}, \underset{A_2}{\frac{6}{21}}, \underset{A_3}{\frac{9}{63}}, \frac{12}{189}, \dots\right)$

② $\left(\underset{B_1}{\frac{3}{2}}, \underset{B_2}{\frac{8}{3}}, \underset{B_3}{\frac{15}{4}}, \underset{B_4}{\frac{24}{5}}, \frac{35}{6}, \dots\right)$

③ $(\underset{C_1}{20}, \underset{C_2}{1}, \underset{C_3}{21}, \underset{C_4}{22}, \underset{C_5}{43}, \underset{C_6}{65}, 108, \dots)$

1A for $k \geq 1$, $A_k = \frac{3k}{7 \cdot 3^{k-1}}$

1B $A_1 = \frac{3}{7}$, for $\cancel{k} \geq 2$, $A_{\cancel{k}} = A_{n-1} - \frac{3}{21} = A_{n-1} - \frac{1}{7}$

Could also write $A_{n+1} = A_n - \frac{1}{7}$

only works for first 3